

THE GRAPH RAMSEY NUMBER $R(F_\ell, K_6)$

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ABSTRACT. For a given pair of two graphs (F, H) , let $R(F, H)$ be the smallest positive integer r such that for any graph G of order r , either G contains F as a subgraph or the complement of G contains H as a subgraph. Baskoro, Broersma and Surahmat (2005) conjectured that

$$R(F_\ell, K_n) = 2\ell(n-1) + 1$$

for $\ell \geq n \geq 3$, where F_ℓ is the join of K_1 and ℓK_2 . In this paper, we prove that this conjecture is true for the case $n = 6$.

1. INTRODUCTION

Throughout this paper, all graphs are finite and simple. For a given pair of two graphs (F, H) , let $R(F, H)$ be the smallest positive integer r such that for any graph G of order r , G contains F as a subgraph or the complement of G contains H as a subgraph. In general, it is quite difficult to calculate the exact values of $R(F, H)$. However, for sparse graphs F and H , there are many results on the exact values of $R(F, H)$. In this paper, we consider fan graph F_ℓ , which is the join of K_1 and ℓK_2 . This fan graph is one example of such sparse graphs.

Recently, although K_n itself is the densest graph, some authors succeeded in determining the value $R(F_\ell, K_n)$ for large ℓ and small n . As for this problem, Baskoro, Broersma and Surahmat [1] conjectured:

Conjecture (The Baskoro-Broersma-Surahmat conjecture [1]).

For any $\ell \geq n \geq 3$, we have $R(F_\ell, K_n) = 2\ell(n-1) + 1$.

In the last 20 years, this conjecture was proved to be true for the case $n = 3, 4, 5$:

Theorem A (Gupta, Gupta and Sudan [5], Li and Rousseau [6, Proposition 1]).

For any $\ell \geq 3$, we have $R(F_\ell, K_3) = 4\ell + 1$.

Theorem B (Baskoro, Broersma and Surahmat [1]).

For any $\ell \geq 4$, we have $R(F_\ell, K_4) = 6\ell + 1$.

Theorem C (Chen and Zhang [2]).

For any $\ell \geq 5$, we have $R(F_\ell, K_5) = 8\ell + 1$.

In this paper, we prove the Baskoro-Broersma-Surahmat conjecture for $n = 6$.

Theorem 1.

For any $\ell \geq 6$, we have $R(F_\ell, K_6) = 10\ell + 1$.

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Our main strategy is based on the approach of Chen and Zhang [2]. In particular, we shall heavily use their structural setting. However, we need some additional insight on the Chen-Zhang structure than their paper [2].

We briefly summarize our notation and terminology. Let $G = (V, E)$ be a graph. For any subset $S \subset V$, we use $G[S]$ to denote the subgraph induced by S . For any vertex $v \in V$, we denote the neighborhood of v by $N(v)$, i.e.

$$N(v) = \{ w \in V \mid vw \in E, w \neq v \},$$

and we let $N[v] = N(v) \cup \{v\}$. We let

$$d(v) = |N(v)|, \quad \delta(G) = \min_{v \in V} d(v), \quad \Delta(G) = \max_{v \in V} d(v).$$

Moreover, we denote by $\omega(G)$ the clique number of G , i.e. the order of the largest clique in G , and denote by $\alpha(G)$ the independence number of G , i.e. the order of the largest independent set in G . We refer to the book [4] for other graph theoretical notation and terminology not described in this paper.

2. THE LOWER BOUND

The lower bound of $R(F_\ell, K_6)$ is given by the following theorem, which is just a special case of the Chvátal-Harary lemma [3, Lemma 4].

Theorem 2. *For any $\ell \geq 6$, we have $R(F_\ell, K_6) \geq 10\ell + 1$.*

Proof. It is sufficient to give a graph G of order 10ℓ such that G does not contain F_ℓ and \overline{G} does not contain K_6 . For example, the graph $G = 5K_{2\ell}$ satisfies this condition. Hence the theorem follows. \square

Hence we prove the upper bound $R(F_\ell, K_6) \leq 10\ell + 1$ in the following sections.

Theorem 3. *For any $\ell \geq 6$, we have $R(F_\ell, K_6) \leq 10\ell + 1$.*

We prove this upper bound by contradiction. In the remaining part of this paper, we assume $\ell \geq 6$ and that there exists a graph G of order $10\ell + 1$ such that $F_\ell \not\subset G$ and $K_6 \not\subset \overline{G}$. By Theorem C, we may assume that $\alpha(G) = 5$.

3. PRELIMINARY LEMMAS

In this section, we prove several lemmas on basic properties of the graph G . We start with the following simple observation, which is related to the number of independent edges in G .

Lemma 1. *There is no vertex $v \in G$ for which $G[N(v)]$ contains ℓK_2 .*

Proof. If $\ell K_2 \subset G[N(v)]$, then $F_\ell \subset G[N[v]]$. This is a contradiction. \square

The next lemma is a special case of Stahl's lemma [7].

Lemma 2. *For any $\ell \geq 1$, we have $R(\ell K_2, K_6) = 2\ell + 4$.*

Proof. See [7, pp. 586–587]. \square

As a consequence of Lemma 2 and $\alpha(G) = 5$, any subgraph $H \subset G$ of order $n \geq 6$ contains $\lfloor n/2 \rfloor - 2$ or more independent edges. The following is an immediate consequence of Theorem C and Lemma 2.

Lemma 3. *We have $2\ell \leq \delta(G) \leq \Delta(G) \leq 2\ell + 3$.*

Proof. First we show that $\delta(G) \geq 2\ell$. Assume, to the contrary, that there exists a vertex v with $d(v) \leq 2\ell - 1$. Let $S = V(G) \setminus N[v]$. Then $|S| \geq 8\ell + 1$. Also, since $\alpha(G) = 5$, it follows that $\alpha(G[S]) \leq 4$. Then $G[S]$ contains F_ℓ by Theorem C, a contradiction. To show that $\Delta(G) \leq 2\ell + 3$, suppose that $d(v) \geq 2\ell + 4$ for some vertex v . Since $\alpha(G) = 5$, Lemma 2 guarantees the existence of ℓK_2 in $G[N(v)]$, producing F_ℓ in G . This cannot occur and so $\Delta(G) \leq 2\ell + 3$, as claimed. \square

The next lemma estimates the clique number $\omega(G)$.

Lemma 4. *We have $\omega(G) \leq 2\ell - 2$.*

Proof. Assume, to the contrary, that G contains a clique H of order $2\ell - 1$. Select a vertex $v_0 \in V(G) \setminus V(H)$ such that

$$|N(v_0) \cap V(H)| = \max\{|N(v) \cap V(H)| \mid v \in V(G) \setminus V(H)\}.$$

The graph $G - H - v_0$ is of order $8\ell + 1$ so that $\alpha(G - H - v_0) = 5$ by Theorem C. Let U be a 5-set of independent vertices in the graph $G_0 = G - H - v_0$.

Since $U \cup \{v\}$ cannot be independent for each $v \in V(G) \setminus U$, there are at least $2\ell - 1$ edges between H and U . Hence, there exists a vertex $u_0 \in U$ with $|N(u_0) \cap V(H)| \geq (2\ell - 1)/5 > 2$. Consequently, $|N(v_0) \cap V(H)| \geq |N(u_0) \cap V(H)| \geq 3$.

Next we show that, for each vertex $v \in V(G_0)$, if $w \in N(v_0) \cap N(v) \cap V(H)$, then $N(v) \cap V(H) = \{w\}$. If this is not the case, say there exists a vertex $w' \in N(v) \cap V(H)$ with $w \neq w'$, then $(\ell - 1)K_2$ in $G[(V(H) \setminus \{w, w'\}) \cup \{v_0\}]$ and the edge vw' form ℓK_2 in $G[N(w)]$, which is impossible.

Now, let $S_1 = N(v_0) \cap V(H) = \{w_1, w_2, \dots, w_t\}$, where $t = |N(v_0) \cap V(H)|$. Then by the above observations, we can find a t -set $U_1 = \{u_1, u_2, \dots, u_t\} \subset U$ such that $N(u_i) \cap V(H) = \{w_i\}$ for $1 \leq i \leq t$. Also, let $S_2 = V(H) \setminus S_1$ and $U_2 = U \setminus U_1$. Recall that $|N(u_0) \cap V(H)| \geq 3$, so $u_0 \in U_2$. Hence, $t = 3, 4$.

Note that there is no edge between U_1 and S_2 and also note that $U \cup \{v\}$ cannot be independent for each $v \in S_2$. Thus there are at least $|S_2| = 2\ell - 1 - t$ edges between U_2 and S_2 . However then,

$$\frac{11 - t}{5 - t} \leq \frac{2\ell - 1 - t}{|U_2|} \leq \max_{u \in U_2} |N(u) \cap V(H)| \leq |N(v_0) \cap V(H)| = t,$$

which cannot occur. This completes the proof. \square

4. STRUCTURAL OBSERVATION

In this section, we give some observation on the structure of the graph G following the argument of Chen and Zhang [2]. Let

$$U = \{u_1, u_2, u_3, u_4, u_5\} \subset V(G)$$

be a 5-set of independent vertices and let

$$X_i := \{v \in V(G) \mid |N(v) \cap U| = i\}$$

for $1 \leq i \leq 5$. Since $\alpha(G) \leq 5$, we have a partition

$$V(G) = U \sqcup X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4 \sqcup X_5.$$

4.1. **On the sets X_i .** Obviously, we have

$$(1) \quad \sum_{i=1}^5 |X_i| = 10\ell - 4$$

and, since $d(u_i) \leq \Delta(G) \leq 2\ell + 3$ by Lemma 3, we have

$$(2) \quad \sum_{i=1}^5 i|X_i| = \sum_{i=1}^5 d(u_i) \leq 10\ell + 15.$$

We will use (1) and (2) throughout this paper. For example, by (1) and (2) we have

$$(3) \quad |X_1| \geq \sum_{i=1}^5 (2-i)|X_i| + 3|X_5| \geq 10\ell - 23 + 3|X_5|,$$

which implies $|X_1| \geq 10\ell - 23$. Also, we have

$$(4) \quad |X_2| \geq \sum_{i=1}^5 (3-i)|X_i| - 2|X_1| \geq 20\ell - 27 - 2|X_1|.$$

Let $I = \{1, 2, 3, 4, 5\}$. For each pair $i, j \in I$, we define the set $X_{ij} = X_i \cap N(u_j)$. Then we have a partition

$$X_1 = X_{11} \sqcup X_{12} \sqcup X_{13} \sqcup X_{14} \sqcup X_{15}.$$

We next find that each X_{1i} induces a clique in G .

Lemma 5. *For each $i \in I$, the graph $G[X_{1i}]$ is complete. Consequently, we have*

$$|X_{1i}| \leq 2\ell - 3, \quad |X_1| \leq 10\ell - 15.$$

Proof. Let $v, v' \in X_{1i}$. Since the 6-set $(U \setminus \{u_i\}) \cup \{v, v'\}$ cannot be independent, it follows that v and v' are adjacent. Therefore, $G[X_{1i} \cup \{u_i\}]$ is a complete graph of order $|X_{1i}| + 1 \leq \omega(G) \leq 2\ell - 2$. In addition, $|X_1| = \sum_{i=1}^5 |X_{1i}| \leq 10\ell - 15$. \square

Assume, without loss of generality, that

$$|X_{11}| \geq |X_{12}| \geq |X_{13}| \geq |X_{14}| \geq |X_{15}|$$

in the rest of our discussion. Thus (3) implies that

$$(5) \quad |X_{15}| = |X_1| - \sum_{i=1}^4 |X_{1i}| \geq 2\ell - 11 \geq 1.$$

In particular, each X_{1i} is nonempty. The next lemma on the sets X_{1i} is immediate yet useful. We will use this lemma mainly with the choice $t = 2$.

Lemma 6. *Let $i, j, k \in I$ be distinct indices and $v \in X_{1i}$. Suppose that $|X_{1i}| \geq 2\ell - 2t$ for some integer with $2 \leq t \leq 6$. Then we have $|N(v) \cap (X_{1j} \cup X_{1k})| \leq 2t$ and $|N(v) \cap X_{1j}| \leq 2t - 1$. In addition, if $|N(v) \cap X_{1j}| \geq 2$ also holds, then we have $|N(v) \cap X_{1k}| \leq 2t - 3$.*

Proof. Note that $(\ell - t)K_2 \subset G[(X_{1i} \setminus \{v\}) \cup \{u_i\}]$. Thus, we must avoid tK_2 in $G[N(v) \cap (X_{1j} \cup X_{1k})]$ and so the result is immediate as each of X_{1j} and X_{1k} induces a clique. \square

4.2. On the sets $N(u_i) \setminus X_{1i}$. Besides the same kind of information as Chen and Zhang obtained, we need some new information on their structural setting. We start with some observations on the set $N(u_i) \setminus X_{1i}$. By Lemma 3 and Lemma 5

$$|N(u_i) \setminus X_{1i}| = d(u_i) - |X_{1i}| \geq 3.$$

In general, suppose that A and B are disjoint nonempty sets of vertices in a graph. Then it is well-known that, if $|N(v) \cap B| \geq |A|$ for every $v \in A$, then there are at least $|A|$ independent edges between A and B . The following is then immediate.

Lemma 7. *Let $i \in I$ and $t = 2\ell - |X_{1i}|$. Every t -set $S \subset N(u_i) \setminus X_{1i}$ contains a vertex v for which $|N(v) \cap X_{1i}| < t$.*

Proof. If $S = \{v_1, v_2, \dots, v_t\} \subset N(u_i) \setminus X_{1i}$ and $|N(v_j) \cap X_{1i}| \geq t$ for $1 \leq j \leq t$, then X_{1i} contains t distinct vertices w_1, w_2, \dots, w_t such that $v_j w_j \in E(G)$ for $1 \leq j \leq t$. However then, $(\ell - t)K_2$ in $G[X_{1i} \setminus \{w_1, w_2, \dots, w_t\}]$ with the t edges $v_j w_j$ ($1 \leq j \leq t$) forms ℓK_2 in $G[N(u_i)]$, a contradiction. \square

Let $i \in I$. Under certain conditions, there exists a 4-set $Q_i \subset V(G)$ satisfying

(6) $Q_i \subset N(u_i) \setminus X_{1i}$ and $Q_i \cup \{w\}$ is an independent 5-set for every $w \in X_{1i}$.

For example, such Q_i exists if the degree of u_i equals $2\ell + 3$, which we verify next.

Lemma 8. *Let $i \in I$. If $d(u_i) = 2\ell + 3$, then there exists a 4-set Q_i satisfying (6).*

Proof. Note that $|N(u_i) \setminus X_{1i}| = d(u_i) - |X_{1i}| \geq 6$ since $d(u_i) = 2\ell + 3$.

Suppose first that $|X_{1i}|$ is odd, say $|X_{1i}| = 2t - 1$ for some positive integer t . Then $G[N(u_i) \setminus X_{1i}]$ contains $(\ell - t)K_2$ by Lemma 2 with four remaining vertices v_1, v_2, v_3, v_4 . Let $Q_i = \{v_1, v_2, v_3, v_4\}$. Then for any $w \in X_{1i}$, $G[X_{1i} \setminus \{w\}]$ contains $(t - 1)K_2$ as $G[X_{1i}]$ is complete. Thus $Q_i \cup \{w\}$ must be independent in order to avoid ℓK_2 in $G[N(u_i)]$.

Suppose next that $|X_{1i}| = 2t$ for some positive integer t . Then $G[N(u_i) \setminus X_{1i}]$ contains $(\ell - t - 1)K_2$ again by Lemma 2 with five remaining vertices v_1, v_2, \dots, v_5 . Since $G[X_{1i}]$ obviously contains tK_2 , the set $Q = \{v_1, v_2, \dots, v_5\}$ must be independent to avoid ℓK_2 in $G[N(u_i)]$. For any $w \in X_{1i}$, $|N(w) \cap Q| \geq 1$ since $Q \cup \{w\}$ cannot be independent. Also, there cannot be two or more independent edges between X_{1i} and Q to avoid ℓK_2 in $G[N(u_i)]$. Thus we can find a vertex in Q , say v_5 , such that every $w \in X_{1i}$ is adjacent to v_5 . Let $Q_i = \{v_1, v_2, v_3, v_4\}$. Recall that $|X_{1i}| = 2t \geq 2$. Then $Q_i \cup \{w\}$ must be independent for every $w \in X_{1i}$ to avoid two independent edges between X_{1i} and Q . \square

4.3. On the sets X_2 and X_3 . We next study the sets X_2 and X_3 in more detail. We start with the following lemma, which can be seen as an analogue of Lemma 5.

Lemma 9. *Let $i, j, k \in I$ be distinct indices.*

- (a) *If $a \in X_{1i}$, $b \in X_{1j}$, $c \in X_{2i} \cap X_{2j}$, then the set $\{a, b, c\}$ is not independent.*
- (b) *If $a \in X_{1i}$, $b \in X_{1j}$, $c \in X_{1k}$, $d \in X_{3i} \cap X_{3j} \cap X_{3k}$, then the set $\{a, b, c, d\}$ is not independent.*

Proof. First, (a) is immediate since the 6-set $(U \setminus \{u_i, u_j\}) \cup \{a, b, c\}$ cannot be independent. Similarly, (b) holds by considering the 6-set $(U \setminus \{u_i, u_j, u_k\}) \cup \{a, b, c, d\}$. \square

The next lemma is an analogue of Claim 1 of Chen and Zhang [2].

Lemma 10. *Let $i, j, k \in I$ be distinct indices.*

- (a) *If $|X_{1i}| = |X_{1j}| = 2\ell - 3$, then $X_{2i} \cap X_{2j} = \emptyset$.*
- (b) *If $|X_{1i}| = |X_{1j}| = |X_{1k}| = 2\ell - 3$, then $X_{3i} \cap X_{3j} \cap X_{3k} = \emptyset$.*

Proof. Let us begin with (a) by assuming, to the contrary, that $c \in X_{2i} \cap X_{2j}$. Since $G[N(c)]$ does not contain ℓK_2 while $G[X_{1i} \cup \{u_i\}] \cong G[X_{1j} \cup \{u_j\}] \cong K_{2\ell-2}$, it follows that $|N(c) \cap (X_{1i} \cup X_{1j})| \leq 2\ell - 2$. Hence, $|(X_{1i} \cup X_{1j}) \setminus N(c)| \geq 2\ell - 4$. Without loss of generality, assume that $|X_{1i} \setminus N(c)| \geq (2\ell - 4)/2 = \ell - 2 \geq 4$. Note that $X_{1j} \setminus N(c) \neq \emptyset$ as $\omega(G) \leq 2\ell - 2$. Let $b \in X_{1j} \setminus N(c)$. By Lemma 9 (a) then, b is adjacent to every vertex in $X_{1i} \setminus N(c)$, i.e. $|N(b) \cap X_{1i}| \geq 4$, contradicting Lemma 6. This proves the assertion (a).

For (b), suppose that $d \in X_{3i} \cap X_{3j} \cap X_{3k}$. Then by a similar argument done in (a), one sees that $|(X_{1i} \cup X_{1j} \cup X_{1k}) \setminus N(d)| \geq 4\ell - 7$ in order to avoid ℓK_2 in $G[N(d)]$. Without loss of generality, suppose that $|X_{1i} \setminus N(d)| \geq |X_{1j} \setminus N(d)| \geq |X_{1k} \setminus N(d)|$. Then

$$6 \leq \lceil (4\ell - 7)/3 \rceil \leq |X_{1i} \setminus N(d)| \leq |X_{1i}| \leq 2\ell - 3$$

and

$$|X_{1j} \setminus N(d)| \geq (4\ell - 7 - |X_{1i}|)/2 \geq \ell - 2 \geq 4.$$

Also, $X_{1k} \setminus N(d) \neq \emptyset$ in order to avoid $K_{2\ell-1}$ in G . Let

$$\{a_1, a_2, \dots, a_6\} \subset X_{1i} \setminus N(d), \quad \{b_1, b_2, b_3, b_4\} \subset X_{1j} \setminus N(d) \quad \text{and} \quad c \in X_{1k} \setminus N(d).$$

By Lemma 6, none of $|N(c) \cap X_{1i}|, |N(c) \cap X_{1j}|, |N(b_1) \cap X_{1i}|$ exceeds 3. By this fact with Lemma 9 (b), we may assume that $b_1 c \notin E(G)$, and $N(c) \cap X_{1i} = \{a_1, a_2, a_3\}$. Then again by Lemma 6, at most one of b_2, b_3, b_4 is adjacent to c and so suppose that $b_2 c \notin E(G)$. Thus, by Lemma 9, $N(b_1) \cap X_{1i} = N(b_2) \cap X_{1i} = \{a_4, a_5, a_6\}$. This gives us $(\ell - 2)K_2$ in $G[(X_{1j} \setminus \{b_1, b_2\}) \cup \{u_j\}]$ with two edges $a_4 a_5$ and $a_6 b_2$ in $G[N(b_1)]$, which is impossible. \square

We let

$$J := \{i \in I \mid \text{a subset } Q_i \subset N(u_i) \setminus X_{1i} \text{ satisfying (6) exists}\}.$$

The next lemma describes how the sets Q_i intersect each other.

Lemma 11.

- (a) *Let $i, j \in J$ be indices with $i < j$. If $|X_{1i}| + |X_{1j}| \geq 2\ell - 1$, then we have $Q_i \cap Q_j \cap X_2 = \emptyset$. Consequently, if $|X_{14}| + |X_{15}| \geq 2\ell - 1$, then*

$$\sum_{i \in J} |Q_i \cap X_2| \leq |X_2|.$$

- (b) *Let $i, j, k \in J$ be indices with $i < j < k$. If $|X_{1i}| + |X_{1j}| + |X_{1k}| \geq 6\ell - 14$, then $Q_i \cap Q_j \cap Q_k \cap X_3 = \emptyset$. Consequently, if $|X_{13}| + |X_{14}| + |X_{15}| \geq 6\ell - 14$, then*

$$\sum_{i \in J} |Q_i \cap X_3| \leq 2|X_3|.$$

Proof. For (a), suppose that $c \in Q_i \cap Q_j \cap X_2$. Then no vertex in $X_{1i} \cup X_{1j}$ is adjacent to c by (6), so $G[X_{1i} \cup X_{1j}]$ is a clique by Lemma 9 (a). However then, $\omega(G) \geq |X_{1i}| + |X_{1j}| \geq 2\ell - 1$. As a result, $Q_i \cap Q_j \cap X_2 = \emptyset$.

For (b), suppose that $d \in Q_i \cap Q_j \cap Q_k \cap X_3$. Then no vertex in $X_{1i} \cup X_{1j} \cup X_{1k}$ is adjacent to d by (6). By the assumption $|X_{1i}| + |X_{1j}| + |X_{1k}| \geq 6\ell - 14$, we have

$$2\ell - 4 \leq |X_{1i}| \leq 2\ell - 3, \quad 2\ell - 5 \leq |X_{1j}| \leq 2\ell - 3, \quad 2\ell - 8 \leq |X_{1k}| \leq 2\ell - 3.$$

We consider two cases separately according to whether $|X_{1k}| \leq 2\ell - 7$ or not.

We first consider the case $|X_{1k}| \leq 2\ell - 7$. In this case, we have $|X_{1j}| \geq 2\ell - 4$ by the assumption. Therefore we have $|X_{1i}|, |X_{1j}| \geq 8$. Take a vertex $c \in X_{1k}$. By Lemma 6 with $t = 4$, we find that $|N(c) \cap X_{1i}|, |N(c) \cap X_{1j}| \leq 7$. Thus we can take vertices $a \in X_{1i} \setminus N(c)$ and $b \in X_{1j} \setminus N(c)$. By Lemma 6 with $t = 2$, we find that $|N(a) \cap X_{1j}|, |N(b) \cap X_{1i}| \leq 3$, i.e. $|X_{1j} \setminus N(a)|, |X_{1i} \setminus N(b)| \geq 5$. By Lemma 9 (b), this implies $|N(c) \cap (X_{1i} \cup X_{1j})| \geq 10$, contradicting Lemma 6 with $t = 4$.

We next consider the case $|X_{1k}| \geq 2\ell - 6$. In this case, we have $|X_{1i}| \geq 8$, $|X_{1j}| \geq 7$ and $|X_{1k}| \geq 6$. Thus we can take vertices

$$\{a_1, a_2, \dots, a_8\} \subset X_{1i}, \quad \{b_1, b_2, \dots, b_7\} \subset X_{1j} \quad \text{and} \quad c \in X_{1k}.$$

By Lemma 6 with $t = 3$, we find that $|N(c) \cap X_{1i}|, |N(c) \cap X_{1j}|, |N(b_1) \cap X_{1i}| \leq 5$. Thus we may assume that $a_1c, b_1c, a_2b_1, a_3b_1 \notin E(G)$. By Lemma 9 (b), we find that $a_1 \in N(b_1) \cap X_{1i}$ and $a_2, a_3 \in N(c) \cap X_{1i}$. Also, by Lemma 6 with $t = 2$, we see that $|N(a_1) \cap X_{1j}| \leq 3$. Thus we may assume that $a_1b_2, a_1b_3, a_1b_4, a_1b_5 \notin E(G)$. However, again by Lemma 9 (b), we find that $b_2, b_3, b_4, b_5 \in N(c) \cap X_{1j}$, contradicting Lemma 6 with $t = 3$. \square

4.4. On the set X_1 . Based on the above observations, we next give a closer look at X_1 . Recall that $10\ell - 23 \leq |X_1| \leq 10\ell - 15$. The next lemma gives an improved upper bound for $|X_1|$.

Lemma 12. *We have $|X_{14}| \leq 2\ell - 4$. Consequently, $|X_1| \leq 10\ell - 17$.*

Proof. If the statement is false, then $|X_{1i}| = 2\ell - 3$ for $1 \leq i \leq 4$. Thus, $X_2, X_3 \subset N(u_5)$ by Lemma 10. Then $X_{15} \cup X_2 \cup X_3 \cup X_5 \subset N(u_5)$ and so $|X_{15}| + |X_2| + |X_3| + |X_5| \leq d(u_5) \leq 2\ell + 3$. This then implies that

$$10\ell - 4 = \sum_{i=1}^4 |X_{1i}| + (|X_{15}| + |X_2| + |X_3| + |X_5|) + |X_4| \leq 10\ell - 9 + |X_4|,$$

so that $|X_4| \geq 5$. On the other hand, using (1) and (2),

$$2|X_4| \leq \sum_{i=2}^5 (i-2)|X_i| \leq 8,$$

so that $|X_4| \leq 4$. Since this is impossible, it follows that $|X_{14}| \leq 2\ell - 4$ and so $|X_1| \leq 3(2\ell - 3) + 2(2\ell - 4) = 10\ell - 17$. This completes the proof. \square

Lemma 13. *The graph G contains a 5-set of independent vertices*

$$U = \{u_1, u_2, \dots, u_5\} \subset V(G)$$

for which either (A) $|X_1| \in \{10\ell - 18, 10\ell - 17\}$ or (B) $X_5 \neq \emptyset$, i.e. each of the five vertices in U is adjacent to a common vertex.

Proof. Let $U = \{u_1, u_2, \dots, u_5\}$ be a 5-set of independent vertices in G and suppose that U does not satisfy (A). By Lemma 12, we have $|X_1| \leq 10\ell - 19$. Then by (1) and (2), we have

$$\sum_{i=1}^5 d(u_i) = \sum_{i=1}^5 i|X_i| \geq 2 \sum_{i=1}^5 |X_i| - |X_1| \geq 10\ell + 11$$

so that $d(u_i) = 2\ell + 3$ for some $i \in I$. By Lemma 8, therefore, we may assume the existence of a 4-set Q_i that satisfies (6). Select a vertex $v \in X_{1i}$ and consider the

5-set $U' = Q_i \cup \{v\}$, which must be independent. Note also that each of the five vertices in U' is adjacent to u_i . Thus, U' satisfies (B). \square

For the rest of this paper, U denotes a fixed 5-set of independent vertices in G satisfying either (A) or (B) in Lemma 13.

4.5. Additional lemmas when $|X_{13}| = 2\ell - 3$.

We prepare some more lemmas on X_2 under the condition $|X_{13}| = 2\ell - 3$.

Lemma 14. *Suppose that $|X_{13}| = 2\ell - 3$. Then $|X_2| = |X_{24}| + |X_{25}| - |X_{24} \cap X_{25}|$. Furthermore, $|X_{24} \cap X_{25}| \leq 1$.*

Proof. The first assertion is immediate by Lemma 10 (a). By Lemma 10 (a) (b), we see that $X_2, X_3 \subset N(u_4) \cup N(u_5)$. Also, obviously $X_4, X_5 \subset N(u_4) \cup N(u_5)$. Thus

$$\begin{aligned} 4\ell + 6 &\geq |N(u_4)| + |N(u_5)| \geq |X_{14}| + |X_{15}| + |X_{24}| + |X_{25}| + |X_3| + |X_4| + |X_5| \\ &= |X_{24} \cap X_{25}| + \sum_{i=1}^5 |X_i| - (|X_{11}| + |X_{12}| + |X_{13}|) \\ &= |X_{24} \cap X_{25}| + 4\ell + 5, \end{aligned}$$

so that $|X_{24} \cap X_{25}| \leq 1$. This proves the lemma. \square

Lemma 15. *Suppose that $|X_{13}| = 2\ell - 3$ and $v \in X_{2i}$ for some $i \in \{4, 5\}$.*

- (a) *If $v \notin X_{24} \cap X_{25}$, then $|N(v) \cap X_{1i}| \geq |X_{1i}| - 3$.*
- (b) *If $v \notin X_{24} \cap X_{25}$ and $|X_{1i}| = 2\ell - 4$, then $G[X_{1i} \cup \{v\}] \cong K_{2\ell-3}$.*

Proof. Suppose that $v \notin X_{24} \cap X_{25}$, that is, $v \in X_{2j}$ for some $j \in \{1, 2, 3\}$. Since $\omega(G) \leq 2\ell - 2$, there is $w \in X_{1j} \setminus N(v)$. By Lemma 9 (a), observe that w is adjacent to each vertex in $X_{1i} \setminus N(v)$. Thus, $|X_{1i} \setminus N(v)| \leq |N(w) \cap X_{1i}| \leq 3$ by Lemma 6. This proves (a).

We next prove the assertion (b). Assume, to the contrary, that $v \notin X_{24} \cap X_{25}$, $|X_{1i}| = 2\ell - 4$ and $G[X_{1i} \cup \{v\}]$ is not complete. Suppose $v \in X_{2j}$ with $j \in \{1, 2, 3\}$. By the same argument above, we see that $|X_{1i} \setminus N(v)|, |X_{1j} \setminus N(v)| \leq 3$ since we can take some vertex $w' \in X_{1i} \setminus N(v)$ by the assumption. However then, $|N(v) \cap (X_{1i} \cup X_{1j})| \geq |X_{1i}| + |X_{1j}| - 6 = 4\ell - 13 \geq 2\ell - 1$, producing ℓK_2 in $G[N(v) \cap (X_{1i} \cup X_{1j} \cup \{u_i, u_j\})]$. Since this does not occur, (b) also holds. \square

Lemma 16. *Suppose that $|X_{13}| = 2\ell - 3$. If $|X_{1i}| \geq 2\ell - 5$ for some $i \in \{4, 5\}$, then $|X_{1i}| + |X_{2i}| \leq 2\ell$.*

Proof. Note that $|X_{1i}| \in \{2\ell - 5, 2\ell - 4\}$ by Lemma 12.

If $|X_{1i}| = 2\ell - 4$ and $|X_{2i}| \geq 5$, then X_{2i} contains four vertices v_1, v_2, v_3, v_4 such that $|N(v_j) \cap X_{1i}| \geq |X_{1i}| - 3 \geq 5$ for $1 \leq j \leq 4$ by Lemma 15 (a) since $|X_{24} \cap X_{25}| \leq 1$. This however contradicts Lemma 7. Thus, $|X_{2i}| \leq 4$, as desired.

Similarly, if $|X_{1i}| = 2\ell - 5$ and $|X_{2i}| \geq 6$, say $\{v_1, v_2, \dots, v_6\} \subset X_{2i}$, then we may assume that $v_1 v_2 \in E(G)$ as $\alpha(G) = 5$ and $|N(v_j) \cap X_{1i}| \geq |X_{1i}| - 3 \geq 4$ for $3 \leq j \leq 5$. Thus, there are three distinct vertices $w_3, w_4, w_5 \in X_{1i}$ such that $v_j w_j \in E(G)$ for $3 \leq j \leq 5$. These three edges with the edge $v_1 v_2$ and $(\ell - 4)K_2$ in $G[X_{1i} \setminus \{w_3, w_4, w_5\}]$ result in ℓK_2 in $G[N(u_i)]$, again a contradiction. It follows that $|X_{2i}| \leq 5$. \square

5. COMPLETION OF THE PROOF

We are now prepared to prove Theorem 3. Recall that we are now considering a fixed 5-set U of independent vertices in G satisfying one of the conditions

$$(A) |X_1| \in \{10\ell - 18, 10\ell - 17\} \quad \text{or} \quad (B) X_5 \neq \emptyset.$$

In the following subsections, we consider these two cases separately.

5.1. **Case A:** $|X_1| \in \{10\ell - 18, 10\ell - 17\}$.

In this subsection, we consider Case A. In this case, observe that $|X_{11}| = |X_{12}| = 2\ell - 3$ and $|X_{14}| = 2\ell - 4$ since otherwise we have $|X_1| \leq 10\ell - 19$ by Lemma 12. Thus we also have $|X_{13}| + |X_{15}| \in \{4\ell - 8, 4\ell - 7\}$. Recalling (4), we have

$$|X_2| \geq 20\ell - 27 - 2|X_1| = \begin{cases} 9 & (\text{if } |X_{13}| + |X_{15}| = 4\ell - 8), \\ 7 & (\text{if } |X_{13}| + |X_{15}| = 4\ell - 7). \end{cases}$$

We now consider two subcases, according to the value of $|X_{13}|$.

5.1.1. **Subcase A1:** $|X_{13}| = 2\ell - 3$.

In this case, $|X_{15}| \in \{2\ell - 5, 2\ell - 4\}$. Again by (4) with Lemmas 14 and 16,

$$\begin{aligned} |X_2| &\geq 20\ell - 27 - 2|X_1| \\ &= 8\ell - 9 - 2(|X_{14}| + |X_{15}|) \\ (7) \quad &\geq 2(|X_{24}| + |X_{25}|) - 9 \\ &= 2(|X_2| + |X_{24} \cap X_{25}|) - 9. \end{aligned}$$

Thus,

$$(8) \quad 9 - 2|X_{24} \cap X_{25}| \geq |X_2| \geq \begin{cases} 9 & (\text{if } |X_{15}| = 2\ell - 5), \\ 7 & (\text{if } |X_{15}| = 2\ell - 4). \end{cases}$$

First, if $X_{24} \cap X_{25} \neq \emptyset$, then it is immediate by (7) and (8) that $|X_{1i}| = 2\ell - |X_{2i}| = 2\ell - 4 \geq 8$ for $i = 4, 5$. Let $X_{24} \cap X_{25} = \{v\}$. By Lemma 15 (b), each of the three vertices in $X_{2i} \setminus \{v\}$ is adjacent to every vertex in X_{1i} for $i = 4, 5$. Since neither $G[N(u_4)]$ nor $G[N(u_5)]$ contains ℓK_2 , we find that $N(v) \cap (X_{14} \cup X_{15}) = \emptyset$. However then, $G[X_{14} \cup X_{15}] \cong K_{4\ell-8}$ by Lemma 9 (a), which cannot occur since $4\ell - 8 > 2\ell - 2 \geq \omega(G)$.

Thus, we may assume that $X_{24} \cap X_{25} = \emptyset$. By (7) and (8) again, $|X_{1i}| = 2\ell - |X_{2i}| = 2\ell - 4$ for some $i \in \{4, 5\}$. Then Lemma 15 (b) implies that each of the four vertices in X_{2i} is adjacent to every vertex in X_{1i} . Thus, $|N(v) \cap X_{1i}| = |X_{1i}| = 2\ell - 4 \geq 8$ for every $v \in X_{2i}$. However, this contradicts Lemma 7.

5.1.2. **Subcase A2:** $|X_{13}| = 2\ell - 4$.

In this case, $|X_{1i}| = 2\ell - 4$ for $3 \leq i \leq 5$ and $|X_1| = 10\ell - 18$.

We first verify that, if $v \in X_2$, then $v \in X_{2j}$ and $|N(v) \cap X_{1j}| \geq 5$ for some $j \in \{3, 4, 5\}$. To see this, suppose that $v \in X_{2i} \cap X_{2j}$ with $1 \leq i < j \leq 5$. If $i \leq 2$, then $3 \leq j \leq 5$ as $X_{21} \cap X_{22} = \emptyset$ by Lemma 10. Since $\omega(G) \leq 2\ell - 2$ and $|X_{1i}| = 2\ell - 3$, there is a vertex $w \in X_{1i}$ such that $vw \notin E(G)$. Then w is adjacent to every vertex in $X_{1j} \setminus N(v)$ by Lemma 9 (a), implying that $|X_{1j} \setminus N(v)| \leq 3$ by Lemma 6. If $3 \leq i < j \leq 5$, on the other hand, then a similar reasoning shows that $|X_{1i} \setminus N(v)| \leq 3$ or $|X_{1j} \setminus N(v)| \leq 3$. Consequently, $|X_{1j} \setminus N(v)| \leq 3$ for some $j \in \{3, 4, 5\}$ in each case, i.e. $|N(v) \cap X_{1j}| \geq |X_{1j}| - 3 \geq 5$.

We next verify that $|X_2| = 9$. If this is not the case, then $|X_2| \geq 10$. By the pigeonhole principle, there exists an integer $j \in \{3, 4, 5\}$ and a 4-set $S \subset X_{2j}$ such that $|N(v) \cap X_{1j}| \geq 5$ for each $v \in S$, contradicting Lemma 7.

Recalling (4), we have

$$9 = |X_2| \geq |X_2| - |X_4| - 2|X_5| = \sum_{i=1}^5 (3-i)|X_i| - 2|X_1| \geq 9$$

and so $|X_4| = |X_5| = 0$, which further tells us that $|X_3| = 5$. Also, $\sum_{i=1}^5 i|X_i| = 10\ell + 15$, that is, $d(u_i) = 2\ell + 3$ for $1 \leq i \leq 5$. By Lemma 8, we can find five 4-sets Q_1, Q_2, \dots, Q_5 satisfying (6). By Lemma 11 (a) (b) then,

$$(9) \quad 20 = \sum_{i=1}^5 |Q_i| = \sum_{i=1}^5 |Q_i \cap X_2| + \sum_{i=1}^5 |Q_i \cap X_3| \leq |X_2| + 2|X_3| = 19,$$

which is clearly impossible.

As a result, Subcases A1 and A2 are both impossible, i.e. Case A never occurs.

5.2. Case B: $X_5 \neq \emptyset$.

In this subsection, we next consider Case B. In this case, we see that $|X_1| \geq 10\ell - 23 + 3|X_5| \geq 10\ell - 20$ by (3). Since Case A never occurs, we may assume that $|X_1| \in \{10\ell - 20, 10\ell - 19\}$. We again consider two subcases.

5.2.1. Subcase B1: $|X_1| = 10\ell - 20$.

By (3), we find that $|X_3| = |X_4| = 0$ and $|X_5| = 1$, which in turn implies that $|X_2| = 15$. Also, $\sum_{i=1}^5 i|X_i| = 10\ell + 15$ and so $d(u_i) = 2\ell + 3$ for $1 \leq i \leq 5$, which guarantees the existence of five 4-sets Q_1, Q_2, \dots, Q_5 satisfying (6). As done in (9), we can write

$$20 = \sum_{i=1}^5 |Q_i| = \sum_{i=1}^5 |Q_i \cap X_2| + \sum_{i=1}^5 |Q_i \cap X_5| \leq |X_2| + 5 = 20$$

and so $\sum_{i=1}^5 |Q_i \cap X_2| = 15$ and $\sum_{i=1}^5 |Q_i \cap X_5| = 5$. Hence, if we let $X_5 = \{v_0\}$, then each Q_i consists of three vertices in X_2 and v_0 so that $\bigcup_{i=1}^5 Q_i = X_2 \cup X_5$. However then, no vertex in $X_1 \cup X_2$ is adjacent to v_0 as $v_0 \in Q_i$, implying that $N(v_0) = U$. Thus, $d(v_0) = 5 < 2\ell = \delta(G)$, a contradiction.

5.2.2. Subcase B2: $|X_1| = 10\ell - 19$.

First, observe that

$$3 \leq |X_3| + 2|X_4| + 3|X_5| = \sum_{i=1}^5 (i-2)|X_i| + |X_1| \leq 4$$

by (1) and (2). Thus, we find that $|X_4| = 0$, $|X_5| = 1$ and either

$$(i) \ |X_2| = 13 \text{ and } |X_3| = 1 \quad \text{or} \quad (ii) \ |X_2| = 14 \text{ and } |X_3| = 0.$$

We consider these two cases separately. Let $X_5 = \{v_0\}$.

If (i) occurs, then again $\sum_{i=1}^5 i|X_i| = 10\ell + 15$ and so we have 4-sets Q_1, Q_2, \dots, Q_5 satisfying (6). Since $|X_1| = 10\ell - 19$, we can check that $|X_{13}| + |X_{14}| + |X_{15}| \geq 6\ell - 13$.

Thus by Lemma 11 (a) (b),

$$\begin{aligned} 20 &= \sum_{i=1}^5 |Q_i| = \sum_{i=1}^5 |Q_i \cap X_2| + \sum_{i=1}^5 |Q_i \cap X_3| + \sum_{i=1}^5 |Q_i \cap X_5| \\ &\leq |X_2| + 2|X_3| + 5 = 20 \end{aligned}$$

and we arrive at the same contradiction as in Subcase B.1.

Finally, suppose that (ii) occurs. Then $\sum_{i=1}^5 i|X_i| = 10\ell + 14$. Thus, four of the five vertices in U have degree $2\ell + 3$ and the remaining one has degree $2\ell + 2$. Let $i_0 \in I$ for which $d(u_{i_0}) = 2\ell + 2$. We know that there is a 4-set Q_i satisfying (6) for each $i \in I \setminus \{i_0\}$.

In the set $N(u_{i_0}) \setminus X_{1i_0}$, which contains $2\ell + 2 - |X_{1i_0}| \geq 5$ vertices, we show that there exists a 3-set $Q \subset N(u_{i_0}) \setminus X_{1i_0}$ such that $|N(v) \cap X_{1i_0}| \leq 2$ for each $v \in Q$. Recall Lemma 2. First, if $|X_{1i_0}| = 2t - 1$ for some positive integer t , then $G[N(u_{i_0}) \setminus X_{1i_0}]$ contains $(\ell - t - 1)K_2$ and five vertices v_1, v_2, \dots, v_5 . Since $G[N(u_{i_0})]$ cannot contain ℓK_2 , there are at most two independent edges between X_{1i_0} and $\{v_1, v_2, \dots, v_5\}$. Thus, we may assume that $|N(v_j) \cap X_{1i_0}| \leq 2$ for $1 \leq j \leq 3$. Similarly, if $|X_{1i_0}| = 2t$ for some positive integer t , then $G[N(u_{i_0}) \setminus X_{1i_0}]$ contains $(\ell - t - 1)K_2$ and four vertices v_1, v_2, v_3, v_4 . Then there cannot be two or more independent edges between X_{1i_0} and $\{v_1, v_2, v_3, v_4\}$. Hence, we may assume that $|N(v_j) \cap X_{1i_0}| \leq 1$ for $1 \leq j \leq 3$. As a result, we find a 3-set $Q = \{v_1, v_2, v_3\} \subset N(u_{i_0}) \setminus X_{1i_0}$ such that $|N(v) \cap X_{1i_0}| \leq 2$ for each $v \in Q$.

Write $Q'_{i_0} = Q$ and $Q'_i = Q_i$ for each $i \in I \setminus \{i_0\}$. We next verify that $\sum_{i=1}^5 |Q'_i \cap X_2| \leq |X_2|$ by proving that $Q'_i \cap Q'_j \cap X_2 = \emptyset$ for distinct integers $i, j \in I$. First, since $|X_1| = 10\ell - 19$, it follows that $|X_{14}| + |X_{15}| = |X_1| - \sum_{i=1}^3 |X_{1i}| \geq 4\ell - 10 \geq 2\ell + 2$. Hence, by Lemma 11 (a), it suffices to verify the result for $i \in I \setminus i_0$ and $j = i_0$. If $v \in Q'_i \cap Q'_j \cap X_2$, then no vertex in X_{1i} is adjacent to v . Also, we have just shown that v is adjacent to at most two vertices in X_{1j} . Then by Lemma 9 (a), there are at least $|X_{1j}| - 2$ vertices in X_{1j} that are adjacent to every vertex in X_{1i} . Hence, $\omega(G) \geq |X_{1i}| + |X_{1j}| - 2 \geq |X_{14}| + |X_{15}| - 2 \geq 2\ell$, which cannot occur.

Therefore,

$$19 = \sum_{i=1}^5 |Q'_i| = \sum_{i=1}^5 |Q'_i \cap X_2| + \sum_{i=1}^5 |Q'_i \cap X_5| \leq |X_2| + 5 = 19,$$

so v_0 belongs to every Q'_i . Thus, $N[v_0] \subset U \cup Q'_{i_0} \cup X_{1i_0}$. In particular, $|N(v_0) \cap X_{1i_0}| \leq 2$ and so $d(v_0) \leq |U| + |Q'_{i_0} \setminus \{v_0\}| + 2 = 9 < 2\ell$. This is again impossible.

We conclude that Case B never occurs, either.

This completes the proof of Theorem 3 and Theorem 1.

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